Prime-Height Tetration as an Analytic Generator: From x^x to p-Towers via Abel Linearization, Mode Coordinates, and Modular Phase Extraction

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Abstract

We develop a unified analytic framework that extends the "2-tetration" generator x^x to prime-height p-tetration in the user-defined sense:

$$^2x = x^x, \qquad ^3x = x^{x^x}, \qquad ^5x = x^{x^{x^{x^x}}}, \qquad ^7x = x^{x^{x^{x^{x^{x^x}}}}},$$

with right-associative towers. The first contribution is an exact tower calculus: a closed, recursion-based Jacobian ladder for $\frac{d}{dx}(^px)$ and its log-derivative. The second contribution is an Abel-linearization construction in which the nonlinear tower-building operator is conjugated to translation in a continuous height coordinate, yielding a continuous-height super-exponential $T(x,h)$ satisfying $T(x,h+1) = x^{T(x,h)}$ and interpolating integer heights, hence capturing prime heights p as samples $T(x,p) = {}^px$. The third contribution is a mode-coordinate representation for tower values compatible with discrete spectral sums and contour-residue "selector" kernels, together with a modular phase extraction layer that can be applied either to tower values or to linearized height coordinates. Throughout, derivations are exact (non-asymptotic): no numerical approximations are invoked.

Contents

1 Introduction

The function x^x occupies a special role as the simplest nontrivial "self-exponential" generator. In many analytic pipelines, x^x behaves like a nonlinear amplifier: it increases magnitude rapidly, intertwines logarithmic and algebraic structure, and induces nontrivial spectral phases under complex exponentiation. However, x^x is only the height-2 member of a much larger hierarchy of power towers

$$ {}^{n}x \;=\; \underbrace{x^{x^{\cdot^{\cdot^{x}}}}}_{n \text{ copies of } x} $$

where the height n is discrete. In particular, restricting heights to primes $p \in \{2,3,5,7,\dots\}$ selects a sparse but structurally meaningful subsequence of towers.

This paper formalizes three layers that together extend the "2-tetration" theory to prime-height p-tetration:

1. **Tower calculus:** exact recursion identities for tower derivatives and induced measures under change of variables.

2. **Height linearization:** Abel conjugacy that converts tower-building into translation in a continuous height coordinate, defining $T(x,h)$.

3. **Spectral/mode and modular layers:** phase-mode coordinates for tower values and exact residue-comb selectors, plus modular extraction implemented via complex phase integrals.

The resulting framework provides a single analytic object whose integer samples reproduce the tower hierarchy and whose prime-indexed samples define the desired prime-height tetration family.

2 Definitions and Basic Tower Structure

2.1 Right-associative towers and prime-height tetration

We adopt the standard right-associative convention:

$$x^{x^x} = x^{(x^x)}, \qquad x^{x^{x^x}} = x^{(x^{(x^x)})}, \text{ etc.}$$

Define the height-n tower recursively for $n \in \mathbb{N}$:

$$t_1(x) = x, \qquad t_{n+1}(x) = x^{t_n(x)}. \tag{1}$$

Then

$$t_n(x) = {}^n x.$$

Prime-height tetration (in the present sense) is the restriction

$${}^p x := t_p(x) \qquad \text{for prime } p.$$

2.2 Exponential form and log ladder

For $x > 0$ define $\log x$ as the real logarithm unless stated otherwise. Write (1) as

$$t_{n+1}(x) = \exp\bigl(t_n(x) \log x\bigr). \tag{2}$$

Define the log ladder

$$\ell_n(x) := \log(t_n(x)), \tag{3}$$

which satisfies the exact recursion

$$\ell_{n+1}(x) = t_n(x) \log x. \tag{4}$$

Thus (t_n, ℓ_n) form a coupled exact system.

3 Exact Tower Calculus: Derivative and Jacobian Ladders

3.1 Derivative recursion

Differentiate (2) to obtain the exact derivative ladder:

$$t'_{n+1}(x) = t_{n+1}(x)\Bigl(t'_n(x) \log x + \frac{t_n(x)}{x}\Bigr), \qquad x > 0. \tag{5}$$

This identity is purely algebraic/differential and holds wherever the indicated quantities are defined.

3.2 Log-derivative recursion

Define the log-derivative

$$D_n(x) := \frac{t_n'(x)}{t_n(x)}. \tag{6}$$

Dividing (5) by $t_{n+1}(x)$ yields

$$D_{n+1}(x) = t_n'(x)\log x + \frac{t_n(x)}{x}. \tag{7}$$

Using $t_n'(x) = t_n(x)D_n(x)$ and $\ell_{n+1}(x) = t_n(x)\log x$ from (4), we obtain the closed recursion

$$\boxed{D_{n+1}(x) = D_n(x)\,\ell_{n+1}(x) + \frac{t_n(x)}{x}} \qquad \text{with} \qquad D_1(x) = \frac{1}{x}. \tag{8}$$

Consequently,

$$\boxed{t_p'(x) = t_p(x)\,D_p(x)} \qquad \text{for any integer } p \geq 1, \tag{9}$$

and in particular for prime p.

3.3 Explicit anchors for small heights

For reference, the recursion yields:

$$t_2(x) = x^x, \qquad t_2'(x) = x^x(\log x + 1), \qquad D_2(x) = \log x + 1.$$

$$t_3(x) = x^{x^x}, \qquad D_3(x) = x^x\Big((\log x + 1)\log x + \frac{1}{x}\Big), \qquad t_3'(x) = t_3(x)D_3(x).$$

For $p \in \{5, 7, \dots\}$, D_p is determined exactly by (8).

4 Pushforward Measures and Exact Change of Variables

4.1 Tower-induced measure

Let $p \geq 1$ be fixed. On any interval where t_p is invertible, define an inverse branch

$$\tau_p(u) := t_p^{-1}(u),$$

so that $u = t_p(x)$ and $x = \tau_p(u)$. Then

$$du = t_p'(x)\,dx = t_p(x)D_p(x)\,dx = u\,D_p(\tau_p(u))\,dx,$$

hence the exact Jacobian

$$\boxed{dx = \frac{du}{u\,D_p(\tau_p(u))}.} \tag{10}$$

This yields a master pushforward identity:

$$\boxed{\int \Phi\big(t_p(x)\big)\,W(x)\,dx = \int \Phi(u)\,W(\tau_p(u))\,\frac{du}{u\,D_p(\tau_p(u))}} \tag{11}$$

for any integrable Φ and W on the relevant branch domain.

4.2 Tower-Mellin and tower-Laplace templates

Define the tower-Mellin template (where convergent):

$$\mathcal{M}_p(s;\beta) := \int_0^\infty t_p(x)^\beta \, x^{s-1} \, dx.$$

Apply (11) with $\Phi(u) = u^\beta$ and $W(x) = x^{s-1}$:

$$\boxed{\mathcal{M}_p(s;\beta) = \int \frac{u^{\beta-1} \, \tau_p(u)^{s-1}}{D_p(\tau_p(u))} \, du.} \tag{12}$$

Similarly define the tower-Laplace template:

$$\mathcal{L}_p(s) := \int_0^\infty e^{-sx} \, t_p(x) \, dx.$$

With $\Phi(u) = u$ and $W(x) = e^{-sx}$, the factor u cancels against u^{-1} in (10), giving:

$$\boxed{\mathcal{L}_p(s) = \int e^{-s\tau_p(u)} \, \frac{du}{D_p(\tau_p(u))}.} \tag{13}$$

Equations (12)–(13) are exact and constitute the tower analogs of power-law substitutions that produce Gamma factors in simpler settings.

5 Abel Linearization and Continuous-Height Extension

5.1 Tower building as iteration of F_x

Fix $x > 0$ and consider the map

$$F_x(y) := x^y = \exp(y \log x). \tag{14}$$

Then the tower recursion (1) is iteration of F_x starting from 1:

$$t_n(x) = F_x^{\circ n}(1), \qquad n \geq 0, \quad t_0(x) = 1.$$

5.2 Abel equation

An Abel function A_x for F_x is a function satisfying

$$\boxed{A_x(F_x(y)) = A_x(y) + 1,} \qquad \text{i.e.} \qquad A_x(x^y) = A_x(y) + 1. \tag{15}$$

When such A_x exists on a domain and is invertible there, it conjugates the nonlinear map F_x to translation.

5.3 Continuous-height tetration (super-exponential)

Assume A_x is invertible on a suitable domain. Define the continuous-height tetration (super-exponential) as

$$\boxed{T(x,h) := A_x^{-1}\big(A_x(1)+h\big).} \tag{16}$$

Then by (15),

$$\boxed{T(x,h+1) = x^{T(x,h)}.} \tag{17}$$

Moreover,

$$T(x,0) = 1, \qquad T(x,1) = x, \tag{18}$$

and by induction,

$$\boxed{T(x,n) = t_n(x) = {}^{n}x, \qquad n \in \mathbb{Z}_{\geq 0}.} \tag{19}$$

5.4 Prime-height tetration as sampling of $T(x,h)$

With (19), prime-height tetration is simply

$$\boxed{{}^{p}x = T(x,p) \qquad \text{for prime } p.} \tag{20}$$

Thus the analytic object $T(x,h)$ unifies all integer towers and yields the prime subsequence by restriction.

5.5 Super-logarithm (height coordinate)

Define the super-logarithm (height coordinate) as

$$\boxed{\mathrm{slog}_x(y) := A_x(y) - A_x(1).} \tag{21}$$

Then (15) implies

$$\boxed{\mathrm{slog}_x(x^y) = \mathrm{slog}_x(y) + 1,} \tag{22}$$

and by construction,

$$\boxed{\mathrm{slog}_x(T(x,h)) = h, \qquad T(x,h) = \mathrm{slog}_x^{-1}(h).} \tag{23}$$

In this coordinate, the tower-building step is exactly additive.

5.6 Exact height derivative

Differentiate (16) with respect to h:

$$A_x(T(x,h)) = A_x(1) + h \quad \Rightarrow \quad A_x'(T(x,h))\,\partial_h T(x,h) = 1.$$

Hence the exact identity

$$\boxed{\partial_h T(x,h) = \frac{1}{A_x'(T(x,h))}.} \tag{24}$$

6 Mode Coordinates for Tower Values

6.1 Value-mode coordinates

Let $f(x)$ be a positive-valued generator. A generic value-mode coordinate (compatible with complex phases) can be written as

$$\alpha_k[f](x) := \frac{\lambda}{i\pi} \log\left(\frac{f(x)}{e^k}\right), \tag{25}$$

for a scaling parameter $\lambda \neq 0$. Specialize to $f(x) = t_p(x) = {}^p x$:

$$\boxed{\alpha_k^{(p)}(x) := \frac{\lambda}{i\pi} \log\left(\frac{t_p(x)}{e^k}\right).} \tag{26}$$

6.2 Formal phase reconstruction

A phase-mode reconstruction consistent with (26) takes the form

$$\boxed{t_p(x) = \sum_{k\geq 1} \left(e^k\right)^{\frac{i\pi}{\lambda}\alpha_k^{(p)}(x)}.} \tag{27}$$

Equation (27) is a representation scheme: the tower value is encoded into logarithmic phase coordinates $\alpha_k^{(p)}(x)$ and reassembled via exponential modes.

6.3 Spectral sum functional

Define the associated spectral sum functional

$$\boxed{\mathcal{Z}^{(p)}(\alpha, \lambda) := \sum_{k\geq 1} \exp\left(\frac{i\pi}{\lambda}\alpha_k^{(p)}\, k\right).} \tag{28}$$

This is a discrete spectral object whose exact discrete extraction can be enforced by residue-comb kernels.

7 Residue-Comb Selectors via $\pi \cot(\pi z)$

7.1 Cotangent kernel identity

The classical identity for converting sums over integers into contour integrals uses $\pi \cot(\pi z)$ as a residue comb. For functions $H(z)$ analytic in a suitable strip with decay conditions, one has

$$\boxed{\sum_{k\in\mathbb{Z}} H(k) = \frac{1}{2\pi i} \oint_C H(z)\, \pi \cot(\pi z)\, dz,} \tag{29}$$

where C encloses the integers in the desired range and $\pi \cot(\pi z)$ has simple poles at $z \in \mathbb{Z}$ with residue 1.

7.2 Application to tower spectral sums

Apply (29) to

$$H(z) := \exp\left(\frac{i\pi}{\lambda}\alpha_z^{(p)}\, z\right)$$

with an analytic continuation $\alpha_z^{(p)}$ of the mode index. Then

$$\boxed{\sum_{k\geq 1}\exp\left(\frac{i\pi}{\lambda}\alpha_k^{(p)}\, k\right) = \frac{1}{2\pi i}\oint_C \exp\left(\frac{i\pi}{\lambda}\alpha_z^{(p)}\, z\right)\,\pi\cot(\pi z)\,dz.} \tag{30}$$

This furnishes an exact analytic selector mechanism for discrete spectral content associated with prime-height towers.

8 Modular Phase Extraction Layer

8.1 Generic modular extraction functional

Let $m \in \mathbb{N}$, $m \geq 2$. For a real-valued function $f(x)$, define a modular-phase extraction functional

$$\boxed{\mathsf{Mod}_m[f](x) := \frac{m}{2\pi}\arg\left(\int_0^{m-1}\exp\Big(\frac{2\pi i}{m}\big(f(x)+k\big)\Big)\,dk\right).} \tag{31}$$

This construction encodes congruence class information through phase winding in the complex exponential.

8.2 Specialization to prime-height towers

With $f(x) = t_p(x) = {}^{p}x$,

$$\boxed{\mathsf{Mod}_m[t_p](x) = \frac{m}{2\pi}\arg\left(\int_0^{m-1}\exp\Big(\frac{2\pi i}{m}\big(t_p(x)+k\big)\Big)\,dk\right).} \tag{32}$$

This gives a modular observable derived from prime-height tetration.

8.3 Kernels built from modular phase

Let Ψ be a real-valued functional, e.g. $\Psi(\theta) = \cos(\theta)$ or $\Psi(\theta) = \sin(\theta)$. Define the modular-phase kernel

$$\boxed{G_{p,m}(x) := \Psi\Big(\arg\int_0^{m-1}\exp\Big(\frac{2\pi i}{m}(t_p(x)+k)\Big)\,dk\Big)\; g(x),} \tag{33}$$

for a chosen weight $g(x)$. Then integral transforms (Mellin/Laplace) of $G_{p,m}$ are induced exactly, and may be further expressed via the pushforward formulas of Section 4.

9 Modular Extraction in the Linearized Height Coordinate

9.1 Height-domain modular extraction

Because $\operatorname{slog}_x(y)$ is additive under $y \mapsto x^y$ by (22), modular extraction can be applied directly to height:

$$\boxed{\mathsf{Mod}_m[\operatorname{slog}_x](y) := \frac{m}{2\pi} \arg\left(\int_0^{m-1} \exp\left(\frac{2\pi i}{m} \left(\operatorname{slog}_x(y) + k \right) \right) dk \right).} \tag{34}$$

This shifts the modular observable from the rapidly growing value-domain to the linearized height-domain.

9.2 Prime-height selection viewpoint

Since $T(x,p) = {}^p x$ and $\operatorname{slog}_x(T(x,p)) = p$, the prime-height selection problem can be phrased as extracting information about p from $y = {}^p x$ by passing through slog_x:

$$y = {}^p x \quad \Longrightarrow \quad \operatorname{slog}_x(y) = p.$$

Then modular filters on p (e.g. mod m patterns) are realized by (34) with exact translation covariance inherited from Abel linearization.

10 Synthesis: A Unified Prime-Height Analytic Pipeline

We summarize the full pipeline for prime-height tetration:

1. **Prime-height generator:**

$$t_p(x) = {}^p x, \qquad t_{n+1}(x) = x^{t_n(x)}.$$

2. **Exact differential structure:** compute $D_p(x)$ via (8) and obtain $t_p'(x) = t_p(x) D_p(x)$.

3. **Pushforward transforms:** on invertible branches, apply the exact Jacobian (10) and master identity (11) to obtain tower-induced transform formulas (12)–(13).

4. **Continuous-height extension:** solve Abel (15) (on a chosen domain/normalization) and define $T(x,h)$ by (16), ensuring $T(x,p) = t_p(x)$ for primes p.

5. **Linear height coordinate:** define slog_x by (21), satisfying shift covariance (22) and exact inversion (23).

6. **Mode coordinates and selectors:** encode tower values into $\alpha_k^{(p)}(x)$ via (26), build spectral sums (28), and enforce discrete selection using the cotangent residue comb (30).

7. **Modular phase extraction:** apply modular extraction to values using (32) or to linearized height using (34).

Every step above is exact and composable. The only non-uniqueness arises in the analytic continuation/normalization of Abel functions A_x and associated inverse branches; once a branch is fixed, the identities remain exact within that domain.

11 Conclusion

We extended the "2-tetration" generator x^x into a prime-height tetration family ${}^p x$ and embedded it into a continuous-height analytic object $T(x,h)$ via Abel linearization. The framework integrates three exact layers: (i) tower calculus through a closed Jacobian ladder, enabling rigorous pushforward transforms; (ii) a height-linear coordinate system through slog_x, turning nonlinear iteration into translation; and (iii) spectral/mode encodings and residue-comb selectors compatible with modular phase extraction. Prime heights arise not as isolated special cases but as samples of a unified analytic structure.

A Appendix A: Recursion Summary

For $x > 0$:

$$t_1(x) = x, \qquad t_{n+1}(x) = x^{t_n(x)}.$$

$$\ell_n(x) = \log t_n(x), \qquad \ell_{n+1}(x) = t_n(x) \log x.$$

$$t'_{n+1}(x) = t_{n+1}(x) \left(t'_n(x) \log x + \frac{t_n(x)}{x} \right).$$

$$D_n(x) = \frac{t'_n(x)}{t_n(x)}, \qquad D_{n+1}(x) = D_n(x)\, \ell_{n+1}(x) + \frac{t_n(x)}{x}, \quad D_1(x) = \frac{1}{x}.$$

B Appendix B: Abel, Super-Exponential, and Super-Log

With $F_x(y) = x^y$:

$$A_x(F_x(y)) = A_x(y) + 1.$$

$$T(x,h) = A_x^{-1}(A_x(1) + h), \qquad T(x,h+1) = x^{T(x,h)}.$$

$$\operatorname{slog}_x(y) = A_x(y) - A_x(1), \qquad \operatorname{slog}_x(x^y) = \operatorname{slog}_x(y) + 1.$$

$$\operatorname{slog}_x(T(x,h)) = h, \qquad T(x,h) = \operatorname{slog}_x^{-1}(h).$$